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Potts model on Sierpinski carpets and their relation to hypercubic lattices of non-integer dimensionality

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Abstract. We study the critical frontiers and the high- and low-temperature asymptotic behaviour of the Potts model two-site correlation function for several Sierpinski carpets with fractal dimension lying between one and two. We simulate the Sierpinski carpets by means of appropriate hierarchical lattices, obtaining from their basic cells the two graphs (among others) that arise from the anisotropic bond-moving schemes. They correspond to the asymptotic high- and low-temperature dominant terms for the hierarchical lattice two-site correlation function exact result. By taking the infinite limit size of the basic cell these asymptotic behaviours tend to those of the Sierpinski carpets. It is then possible to compare them with similar results obtained from analytical extension to $d<2$ of the hypercubic lattice. Our results indicate that these asymptotic behaviours do not always coincide, suggesting that possibly we may not identify the Sierpinski carpets exhibited by Gefen *et al* as a correct implementation of the analytical extension of hypercubic lattices.

1. Introduction

The study of critical phenomena on fractal objects is a problem of current research (Gefen *et al* 1980, 1981, 1983a, 1984, Havlin *et al* 1983, Suzuki 1983, Bhanot *et al* 1984, 1985). Gefen *et al* (1980) showed that the Ising model on lattices with fractal dimensionality D_f lying between one and two only presents a non-vanishing critical temperature if the order of ramification (Mandelbrot 1982) is infinite. This is one of the reasons why Sierpinski carpets (SC) can be appropriate to investigate critical phenomena in systems of low dimensionality. In the same work, they also showed that the critical exponents of the SC depend not only on D_f and the spin dimension but also on some topological parameters such as the connectivity Q, the lacunarity L, etc (Mandelbrot 1982). Another interesting feature of the fractals is that they are scale invariant but not translation invariant, except that in some special limits they regain this property. Moreover the relation between the SC and the analytical extension of hypercubic lattices, suggested by Gefen *et al* (1983b), is another important point that, no doubt, deserves a better understanding.

In this work we intend to make a quantitative analysis of the q-state Potts ferromagnet on SC. We also calculate the correlation function low- and high-temperature limits of these SC and compare them with those obtained by analytical extension hypercubic

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lattices. The SC are simulated by appropriate hierarchical lattices (HL) (Berker and Ostlund 1979, Griffiths and Kaufman 1982, Melrose 1983a, b, Kaufman and Griffiths 1984, Tsallis 1985, Hu 1986, Hauser and Saxena 1986). The HL are known to provide accurate results for the critical frontiers (Reynolds *et al* 1977, Stinchcombe 1979, Levy *et al* 1980, de Magalhães *et al* 1981, Oliveira and Tsallis 1984, da Silva *et al* 1984). They also yield high- and low-temperature limits which are in excellent agreement with those associated with hypercubic lattices of integer dimensions (Martin and Tsallis 1981, Curado *et al* 1981, Curado and Hauser 1986). We find real space renormalisation, which is exact on the HL (Potts model), to be a suitable technique for our purposes.

This paper is organised as follows. In § 2 we consider a Potts model on appropriately chosen HL that simulate the SC. From the basic cell of each one of these HL we obtain two graphs (which also generate other ones hereafter referred to as M-type and K-type graphs) that contribute to the correlation function high- and low-temperature-behaviour dominant terms. It can be shown that the M- and K-type graphs coincide with those obtained if we make use of the anisotropic bond-moving schemes (Migdal 1975, Kadanoff 1976) on the SC. In § 3 we calculate the critical frontiers of three SC through the use of the HL and those generated by the M- and K-type graphs. We verify quantitatively how these critical frontiers change with q and compare the frontiers coming from the different graphs.

In § 4 the M- and K-type graphs are used to obtain the two-site correlation function high- and low-temperature behaviour of the SC (we assume that these behaviours are the same as those of the appropriate HL) and to compare them with those corresponding to the analytical extension of equal dimension.

Our conclusions are summarised and discussed in § 5.

2. Renormalisation group scheme for the Sierpinski carpet

In order to understand the SC better we will begin with a simple one. The starting point of its construction is a unit square which, in the first step, we divide into b^2 small squares and cut out symmetrically l^2 centre squares. Then, in the next step we divide each of the remaining $b^2 - l^2$ squares and, once more, cut out l^2 squares. The SC is obtained by the exhaustive repetition of this procedure (see figure 1) and its fractal dimension is (Mandelbrot 1982)

$$D_f = \ln(b^2 - l^2)/\ln b. \tag{1}$$

It is possible to consider a Potts model on the SC, considered as a lattice, by putting a q-state Potts variable σ on each lattice site of the 'microscopic' one (see Gefen *et*

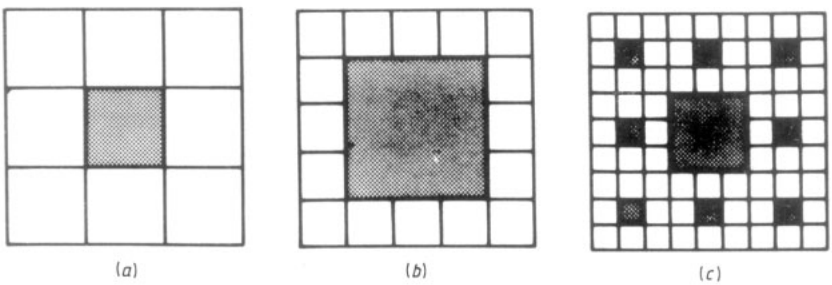


Figure 1. Iteration of the Sierpinski carpet. (a) Initial unit square with $b = 3$, $l = 1$. (b) The first iteration for $b = 5$, $l = 3$. (c) The second iteration for $b = 3$, $l = 1$.

al 1980). The Hamiltonian H is given by

$$H = -qJ \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j} - qJ_{\mathrm{w}} \sum_{\langle l,m \rangle'} \delta_{\sigma_l \sigma_m} \quad (2)$$

where $\langle l, m\rangle'$ means the nearest-neighbour sites that are in the border of the eliminated areas and $\langle i, j\rangle$ means the remaining nearest-neighbour ones.

It has been shown in several works that appropriate hierarchical lattices provide good results to the square lattice (Reynolds *et al* 1977, Stinchcombe 1979, Curado *et al* 1981, Oliveira and Tsallis 1982, Costa and Tsallis 1984) and other hypercubic lattices of integer dimension (da Silva *et al* 1984, Curado and Hauser 1986). Therefore we believe that lattices with non-integer dimensionality, like the SC, can also be well approximated by well chosen HL. We must have two aggregation procedures corresponding to the coupling constants $K = J/k_{\mathrm{B}}T$ and $K_{\mathrm{w}} = J_{\mathrm{w}}/k_{\mathrm{B}}T$ (see Griffiths and Kaufman (1982) for a comment about the aggregation procedure). These are shown in figure 2 for the SC corresponding to $b = 3$, $l = 1$. Thus we have two hierarchical lattices, one simulating the neighbourhood of the interface between two squares as shown in figure 2(a) and the other simulating the neighbourhood of the interface between a square and an eliminated square as in figure 2(c). These hierarchical lattices are shown in figure 3. Note that these lattices are a kind of HL called 'non-uniform' by Griffiths and Kaufman (1982) (see figure 4 in their work). Let us also stress that the 'diamond' lattice with non-iterated bonds exhibited in the same work can be seen as a type of non-uniform hierarchical lattice if we adopt the scheme shown in our

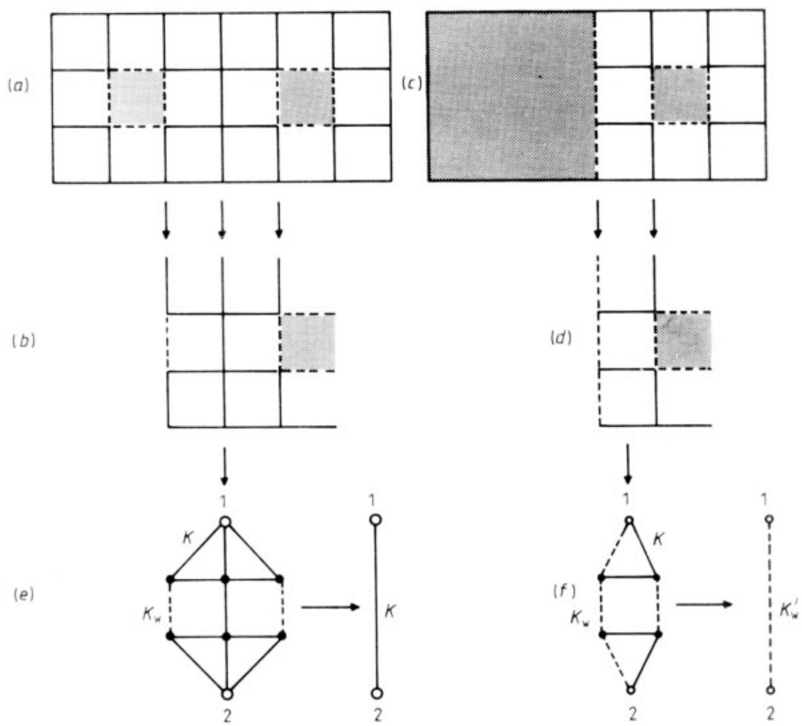


Figure 2. Construction of an HL adequate to simulate the SC with $b = 3$, $l = 1$. (a), (b) and (c) show how K' is obtained. (d), (e) and (f) show how K'_{w} can be performed. Full lines denote the coupling constant K whereas broken lines denote K_{w}.

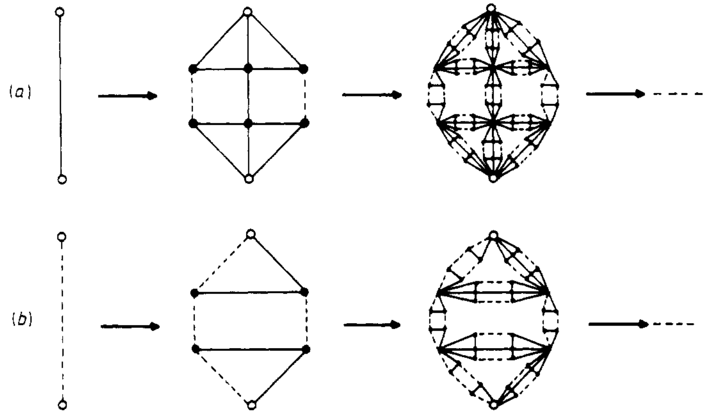


Figure 3. Non-uniform hierarchical lattices corresponding to (*a*) coupling constant K (full lines) and (*b*) K_w (broken lines). The 'internal structure' of the full lines is always as in (*a*) and that of the broken lines as in (*b*).

figure 4. This class of hierarchical lattices mixes two (or more) types of bonds, each with its own aggregation scheme. In other words, the internal structure of a bond contains, besides the same bond, other ones with a different internal structure. The dimensionality and aggregation number of this class of hierarchical lattices are interesting questions that, no doubt, deserve attention. The procedure is similar for other b and l values (other SC). Let us now introduce the variables t and t_w (that are respectively the two-site correlation function of a bond with $J_{ij}=J$ or J_w) as

$$t=\frac{1-\exp(-qJ/k_BT)}{1+(q-1)\exp(-qJ/k_BT)} \tag{3a}$$

$$t_w=\frac{1-\exp(-qJ_w/k_BT)}{1+(q-1)\exp(-qJ_w/k_BT)}. \tag{3b}$$

With these variables, the renormalisation equations (with $q=2$ for simplicity) corresponding to figures 2(*e*) and (*f*), obtained by the break-collapse method (Tsallis and Levy 1981) are (where t' and t'_w are the two-site correlation functions of the graphs

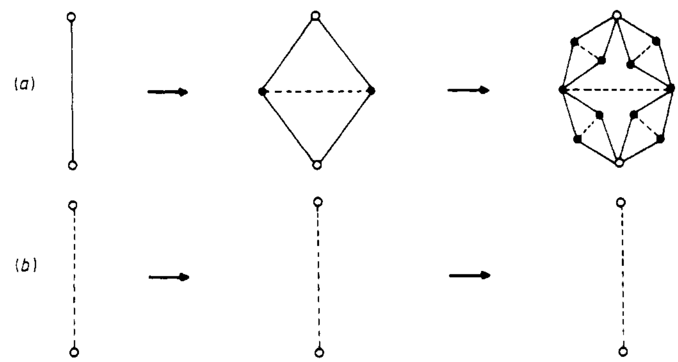


Figure 4. Diamond hierarchical lattice with non-iterated bonds considered as a non-uniform hierarchical lattice corresponding to (*a*) coupling constant K (full lines) and (*b*) K_w (broken lines). Clearly the aggregation scheme (or internal structure) for the full and broken lines are different. The aggregation scheme in (*a*) mixes both lines.

shown in figures 2(e) and (f) respectively):

$$t' = (2t^2t_w + t^3 + 4t^3t_w + 4t^4 + 6t^4t_w + 4t^5 + 8t^5t_w + 4t^5t_w^2 + 6t^6t_w + 8t^6t_w^2 + 2t^7 + 4t^7t_w + 4t^7t_w^2 + 4t^8 + 2t^8t_w + t^{11})(1 + 4t^3 + 2t^3t_w + 2t^4 + 4t^4t_w + 4t^4t_w^2 + 6t^5t_w + 8t^5t_w^2 + 4t^6 + 8t^6t_w + 4t^6t_w^2 + 4t^7 + 6t^7t_w + t^8 + 4t^8t_w + 2t^9t_w)^{-1} \quad (4a)$$

$$t'_w = (t^2t_w + t^4t_w + 4t^2t_w + t_w^3 + t^2t_w^3)(1 + 2t^2t_w + t^2t_w^2 + t^4t_w^2 + 2t^2t_w^3 + t^2t_w^4)^{-1}. \quad (4b)$$

These equations give us the critical frontiers (and critical exponents) corresponding to the case $b = 3$, $l = 1$. It is important to observe that the application of the break-collapse method to solve a graph G always leads to functions $x' = f_G(x)$ where $f_G(x)$ can be written as

$$f_G(x) = N_G(x)/D_G(x) \quad (5)$$

where $N_G(x)$ (and $D_G(x)$) are polynomials of x (see equations (4a) and (4b)). Alternatively, equations (4a) and (4b) can also be written as (for all values of q):

$$t' = [t^4N[G_1] + 4t^3(1-t)N[G_2] + 2t^2(1-t)^2N[G_3] + 2t^2(1-t)^2N[G_4] + 2t^2(1-t)^2N[G_5] + 4t(1-t)^3N[G_6] + (1-t)^4N[G_7]] \times [t^4D[G_1] + 4t^3(1-t)D[G_2] + 2t^2(1-t)^2D[G_3] + 2t^2(1-t)^2D[G_4] + 2t^2(1-t)^2D[G_5] + 4t(1-t)^3D[G_6] + (1-t)^4D[G_7]]^{-1} \quad (6a)$$

$$t'_w = [t^2N[G_8] + 2t(1-t)N[G_9] + (1-t)^2N[G_{10}]] \times [t^2D[G_8] + 2t(1-t)D[G_9] + (1-t)^2D[G_{10}]]^{-1} \quad (6b)$$

where $N[G_i]$ and $D[G_i]$ are the numerator and denominator polynomials, respectively, (in t and t_w) corresponding to graph $[G_i]$ (see figure 5). When $t \to 0$ ($k_BT/J \to \infty$) equations (6a) and (6b) become

$$t' = \frac{N[G_7]}{D[G_7]}\left[1 + 4t\left(\frac{N[G_6]}{N[G_7]} - \frac{D[G_6]}{D[G_7]}\right) + \ldots\right] \quad (7a)$$

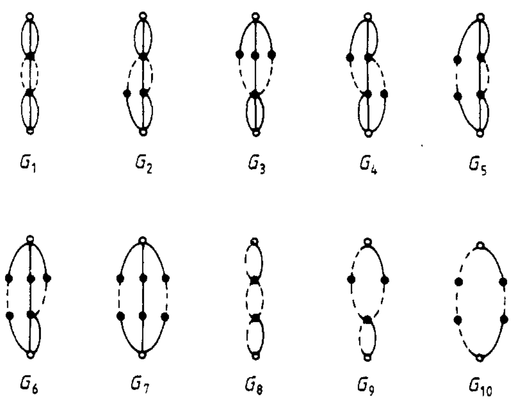


Figure 5. Graphs that arise when we make the break-collapse method. G_1–G_7 on the HL given by figure 2(e) and G_8–G_{10} on the HL given by figure 2(f).

$$t'_{\mathrm{w}} \sim \frac{N[G_{10}]}{D[G_{10}]}\left[1+2t\left(\frac{N[G_9]}{N[G_{10}]}-\frac{D[G_9]}{D[G_{10}]}\right)+\ldots\right] \tag{7b}$$

showing that the graphs G_7 and G_{10} (graphs of M-type) provide the leading term of t' and t'_{w}, respectively, at high temperature. In the case $t \to 1$ ($k_B T/J \to 0$) a similar analysis shows that the dominant terms of t' and t'_{w} are given by the graphs G_1 and G_8, respectively (graphs of K-type). We note that the K-type graphs corresponding to t' and t'_{w} are the same as used by Gefen *et al* (1984) (see their figures 2 and 3), obtained by Kadanoff bond-moving schemes. For other values of b our construction always obtains, in the $t \to 1$ limit, their corresponding graphs. We can note also that the M-type graphs G_7 and G_{10} correspond to the Migdal approximation (Migdal 1975) to figures 2(a) and (c), respectively. The anisotropic bond-moving scheme (x decimation (Kadanoff 1976)) on the clusters of figures 2(b) and (d) provides the K-type graphs corresponding to the renormalisation of the y direction coupling constant. Through a $\pi/2$ rotation in these figures the x decimation now provides the M-type graphs corresponding to the x direction coupling constant.

It is worth observing that our kind of approximation, through HL, always gives a Migdal-type graph (bond-moving schemes adopted by Gefen *et al* (1984)) for the high- (low-) temperature leading term.

3. Potts model critical frontiers to Sierpinski carpets

As pointed out by Gefen *et al* (1984) there are three basic situations for the critical frontiers of the SC defined early in § 2. These situations are: (i) $b=3$, $l=1$, (ii) $b=l+2$, $b>2$, (iii) $b>l+2$. We want to analyse their critical frontiers for several values of q.

In order to compare our results with those of Gefen *et al* (1984) we have studied the frontiers corresponding to the appropriate HL, and to the hierarchical lattices generated by the K-type (HL, $t\to 1$ limit) and M-type (HL, $t \to 0$ limit) graphs. The critical frontiers corresponding to the case $b=3$, $l=1$ are shown in figure 6. In figure 6(a) the HL (schemes represented by figures 2(e) and (f) are used. In figure 6(b) the HL generated by the K-type graphs G_1 and G_8 (that are the same as utilised by Gefen *et al* 1984) are used. Finally in figure 6(c) the M-type graphs G_7 and G_{10} (Migdal type) are used. G_1 and G_7 correspond to t' and G_8 and G_{10} correspond to t'_{w}. As expected, the ordered phase in all of them increases with increasing q. We believe that our HL approach to the SC yields a better approximation than that provided by HL generated by a basic cell of K-type or M-type. As a quantitative example we can see the bond percolation problem ($q=1$) in the SC. The fractal dimension of this SC ($b=3$, $l=1$) is $D_f \simeq 1.893$. Thus it is reasonable to expect that its critical point p_c is slightly greater than the bond-percolation critical point of the square lattice ($p_c=0.5$) because there are now holes in the lattice. In figure 6(a) we can see that the intersection of the $q=1$ critical frontier with the $t=t_{\mathrm{w}}$ line yields $p_c \simeq 0.53$ (which is certainly a good approximation for the exact p_c of the SC bond percolation). The values of p_c from figures 6(b) and (c) are, respectively $p_c \simeq 0.33$ and $p_c \simeq 0.75$. Critical points for other values of q (obtained in a similar way) are shown in table 1. The stability conditions of the fixed points are the same as obtained by Gefen *et al* (1984) (bond-moving schemes, $q=2$) for all values of q and for the two other approximations (see figures 6(a) and (c)). For the localisation of the fixed points E and F (that depend on q) see table 1.

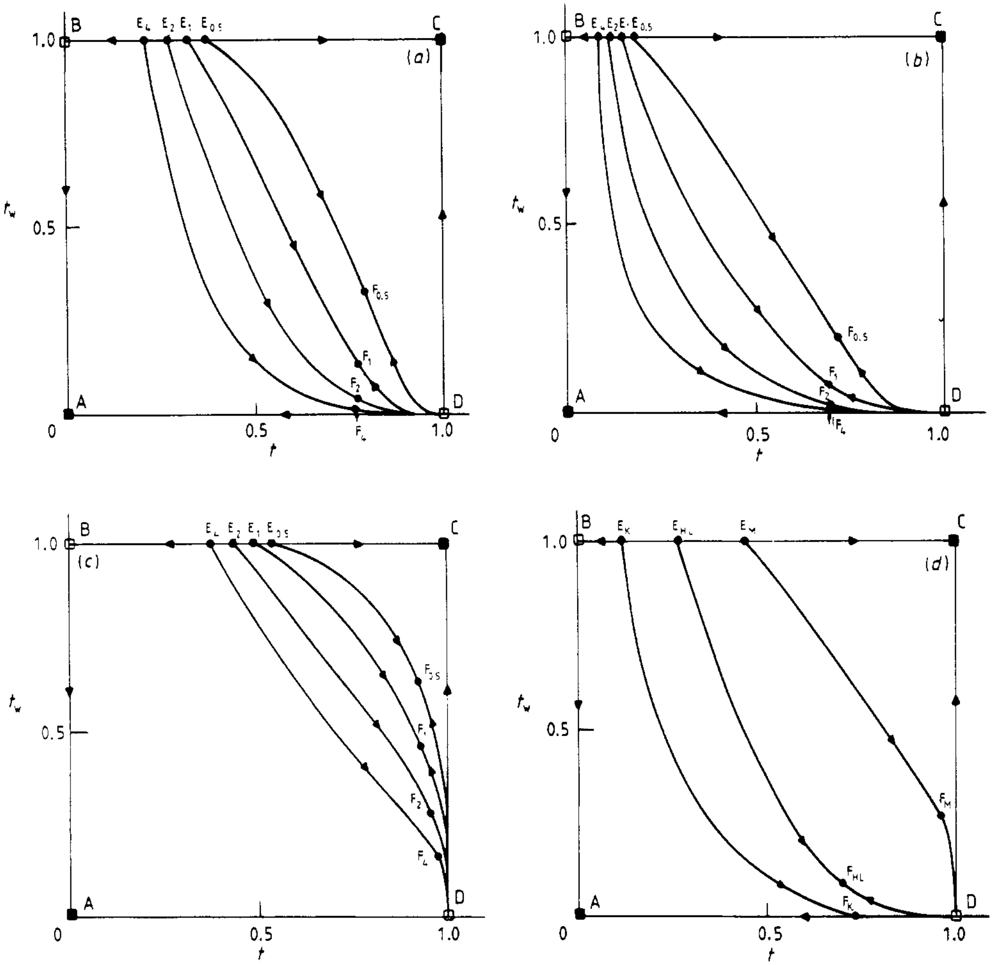


Figure 6. Sierpinski carpet ($b = 3$, $l = 1$) critical frontiers in $t - t_w$ space: (a) through the HL of figure 2, (b) through the corresponding K-type graph, (c) through the corresponding M-type graph, (d) showing the three approximations (HL, K-type and M-type graphs) for $q = 2$. The points A, B, C, D, E and F are fixed points whose stabilities are given by the arrows.

In figure 6(d) we compare the three approximation schemes. We notice that in the low-temperature limit ($t \to 1$) our HL treatment has the same behaviour as the one of the HL associated to the K-type graphs, as was pointed out in § 2.

We have also calculated the phase diagrams for the cases $b = 5$, $l = 3$ and $b = 7$, $l = 3$. Some quantitative results are shown in table 1.

4. Sierpinski carpets and the hypercubic lattices with non-integer dimensionality

Another interesting feature concerning the SC of low lacunarity is their possible relation with the abstract analytic continuation of hypercubic lattices to non-integer dimensionalities, as suggested by Gefen *et al* (1983b).

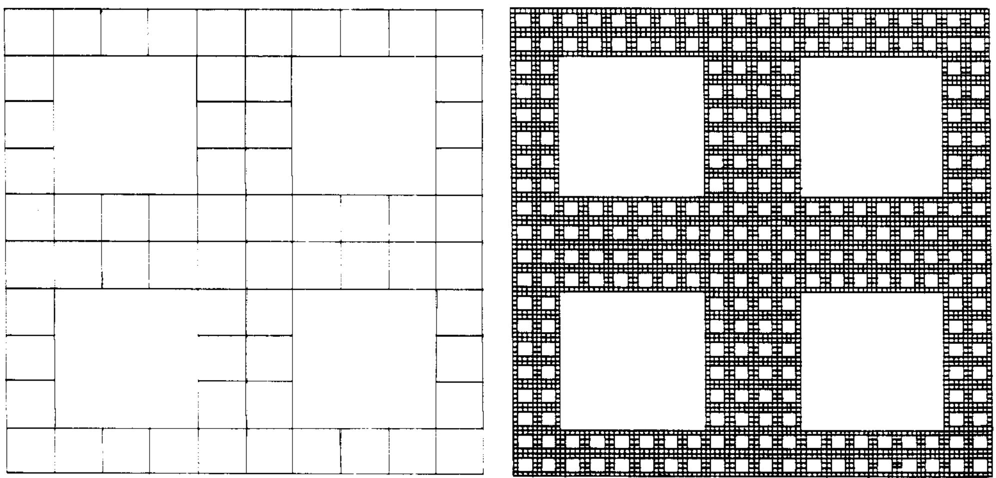
Table 1. Critical points for the Potts model on some Sierpinski carpets obtained by three kinds of approximation (HL, K- and M-type graphs). t_E^*, $(t_F^*, t_{w,F}^*)$ and t^* give, respectively, the localisation of the points E, F and the point obtained by the intersection of the critical frontiers with the $t = t_w$ line.

		$q=\frac{1}{2}$			$q=1$			$q=2$			$q=4$		
		t_E^*	$(t_F^*, t_{w,F}^*)$	t^*	t_E^*	$(t_F^*, t_{w,F}^*)$	t^*	t_E^*	$(t_F^*, t_{w,F}^*)$	t^*	t_E^*	$(t_F^*, t_{w,F}^*)$	t^*
$b=3, l=1$	HL	0.3733	(0.7927, 0.3174)	0.65	0.3224	(0.7718, 0.1354)	0.53	0.2679	(0.7746, 0.0328)	0.45	0.2145	(0.7814, 0.0055)	0.36
$D \simeq 1.893$	K	0.1818	(0.7201, 0.1893)	0.41	0.1516	(0.6990, 0.0683)	0.33	0.1197	(0.6995, 0.0156)	0.23	0.0897	(0.7037, 0.0027)	0.17
	M	0.5379	(0.9266, 0.6232)	0.81	0.4893	(0.9341, 0.4620)	0.75	0.4354	(0.9603, 0.2904)	0.68	0.3795	(0.9838, 0.1612)	0.61
$b=5, l=3$	HL	0.1405	(1, 0.7291)	0.82	0.1269	(1, 0.6106)	0.73	0.1093	(1, 0.4859)	0.62	0.0894	(1, 0.3689)	0.51
$D \simeq 1.723$	K	0.0542	(1, 0.7291)	0.76	0.0486	(1, 0.6106)	0.64	0.0411	(1, 0.4859)	0.52	0.0326	(1, 0.3689)	0.40
	M	0.6238	(1, 0.9472)	0.96	0.5676	(1, 0.9135)	0.94	0.5064	(1, 0.8688)	0.90	0.4432	(1, 0.8154)	0.86
$b=7, l=1$	K	0.1537	(0.4775, 0.0650)	0.39	0.1186	(0.3678, 0.0193)	0.28	0.0866	(0.2682, 0.0045)	0.19	0.601	(0.1866, 0.0009)	0.13
$D \simeq 1.806$	M	0.8263	(0.9315, 0.7048)	0.92	0.7814	(0.9063, 0.5752)	0.89	0.7326	(0.8762, 0.4407)	0.86	0.6817	(0.8408, 0.3187)	0.82

These low-lacunarity SC (see Gefen *et al* 1984) can be constructed as follows. Given a square (of unit area), we divide it into b^2 small squares, each of which is subdivided into c^2 subsquares. In each one of the c^2 squares we cut out l^2 $(l < c)$ subsquares. The procedure is then repeated for the remaining subsquares until one reaches microscopic length scales (see figure 7). The fractal dimension associated with this SC is

$$D_f = \ln[b^2(c^2 - l^2)]/\ln bc. \tag{1'}$$

We associate a q-state Potts variable with each site, a coupling constant K_w with each bond at the border of an eliminated area and the coupling constant K otherwise (nearest neighbour only). Gefen *et al* (1983b) studied the Ising model on these SC and obtained that, in the low-lacunarity limit ($c \to \infty$) and for $c - l$ fixed, the first-order thermal critical exponent is $y = \varepsilon + (\varepsilon^2)$ $(\varepsilon \equiv d - 1)$ which is the same value obtained by analytic continuation of the hypercubic lattices to $d = 1 + \varepsilon$. This and other facts suggest that the SC so constructed could be a geometrical implementation for the analytic extension of hypercubic lattices. In the same work they suggest that other renormalisation group schemes could be used and that in particular the general-dimensional high- and low-temperature expansions of the SC could be compared with that of hypercubic lattices of non-integer dimension. Our alternative renormalisation group scheme (through an appropriate HL) allows us to test the high- and low-temperature expansions utilising the graphs of K- and M-type (that yield the two-site correlation function lower- and higher-temperature behaviour of the HL). The question is that, in the low-lacunarity limit ($c \to \infty$), the size of the HL basic cell goes to infinity and the expansions must describe the exact behaviour of the SC (at least the high- and low-temperature behaviour must be described correctly). Let us say that, intuitively, the contribution of the surface effect caused by the two terminal sites of the HL decreases when the HL basic cell size goes to infinity. In integer dimension this is true (see, e.g., Curado *et al* 1981, Martin and Tsallis 1981, Kaufman and Mon 1984, Curado and Hauser 1986) and there is no reason to believe that this does not happen for the SC (which have non-integer dimensionality).



(*a*) (*b*)

Figure 7. Sierpinski carpet with $b = 2$, $c = 5$ and $l = 3$: (*a*) first iteration, (*b*) second iteration.

That being so, in this section we shall use the HL approximation to the SC of low lacunarity ($c\to\infty$) and to hypercubic lattices and compare them. The M- and K-type graphs, corresponding to the higher- and lower-temperature limits of the HL basic cell constructed to simulate the SC (with b, c and l), have the following recurrence equations for t and t_w.

For K-type graphs:

$$t'=\left(\frac{1-(t^{\rm D})^{bc}}{1+(q-1)(t^{\rm D})^{bc}}\right)^{b(c-l)}\left(\frac{1-(t_{\rm w}^{\rm D})^{2b}(t^{\rm D})^{b(c-l-1)}}{1+(q-1)(t_{\rm w}^{\rm D})^{2b}(t^{\rm D})^{b(c-l-1)}}\right)^{bl} \tag{8a}$$

$$t'_{\rm w}=\left(\frac{1-t_{\rm w}^{\rm D}(t^{\rm D})^{(bc-1)/2}}{1+(q-1)t_{\rm w}^{\rm D}(t^{\rm D})^{(bc-1)/2}}\right)^{b(c-l)}\left(\frac{1-(t_{\rm w}^{\rm D})^{b+1}(t^{\rm D})^{[b(c-l-1)-1]/2}}{1+(q-1)(t_{\rm w}^{\rm D})^{b+1}(t^{\rm D})^{[b(c-l-1)-1]/2}}\right)^{bl}. \tag{8b}$$

For M-type graphs:

$$(t')^{\rm D}=[(t^{bc})^{\rm D}]^{b(c-l-1)}[(t_{\rm w}^{bl}t^{b(c-l)})^{\rm D}]^{2b} \tag{9a}$$

$$(t'_{\rm w})^{\rm D}=[(t^{bc})^{\rm D}]^{[b(c-l-1)-1]/2}[(t_{\rm w}^{bl}t^{b(c-l)})^{\rm D}]^{b}(t_{\rm w}^{bc})^{\rm D}. \tag{9b}$$

Here D means dual and the dual variable is defined as

$$X^{\rm D}\equiv(1-X)[1+(q-1)X]^{-1}. \tag{10}$$

Equations (8) are the same as equations (3) of Gefen *et al* (1983b) (written in a different form). In the high-temperature limit (t, $t_w\to 0$) the leading terms of equations (9) are

$$t'\sim 2bt^{b(c-l)}t_{\rm w}^{bl}+b(c-l-1)t^{bc}+\dots \tag{9a$'$}$$

$$t'_{\rm w}\sim t_{\rm w}^{bc}+bt_{\rm w}^{bl}t^{b(c-l)}+\tfrac{1}{2}[b(c-l-1)-1]t^{bc}+\dots. \tag{9b$'$}$$

We remark that if $c\to\infty$ and $t_w=\frac{1}{2}t$, equations (9) are reduced to one and the same equation.

The two-site correlation function high-temperature behaviour on hypercubic lattices obtained through the HL approximation can be expressed as

$$t'\sim(bc)^{d-1}t^{bc}+{\rm O}(t^{bc+1})\qquad t\to 0. \tag{11}$$

Of course, this expression, obtained for integer values of d, can be analytically extended to $d\in(1,2)$. We want to compare the high-temperature results from the SC (equations (9)) with those obtained from an analytical extension of hypercubic lattices to non-integer dimensions (equation (11)). For simplicity we will compare them in the case $c\to\infty$ and l fixed. From equation (1') we see that the fractal dimension approaches two as

$$D_{\rm f}\sim 2-l^2/c^2\ln(bc)+\dots. \tag{12}$$

If we take the expressions given in equations (9) along the line $t_w=\frac{1}{2}t$ (t, $t_w\to 0$) and consider the limit cited above, then we obtain the expression

$$t'\sim b(c-l-1)t^{bc}+{\rm O}(t^{bc+1}). \tag{13}$$

If, as was claimed, the low-lacunarity SC is an implementation of hypercubic lattices with non-integer dimensionality, the expressions given by equations (11) and (13) must be equal.

Then, matching the coefficients of t^{bc} and taking the limit, we find

$$d\sim 2-(l+1)/c\ln(bc)+\dots. \tag{14}$$

The imposition of equality among the correlation function high-temperature-behaviour dominant terms of the SC and hypercubic lattices of non-integer dimensionality leads to a value of d (given by equation (14)) unlike from the fractal dimension D_f of the SC (given by equation (12)). The low-temperature-behaviour dominant term of the SC two-site correlation function in the same limit ($D_f \to 2$) yields the following expression for d:

$$d \sim 2 - l/c \ln(bc) + \dots \quad (15)$$

which is slightly different from equation (14) and also different from the fractal dimension D_f (equation (12)). So we can conclude that the correlation function higher- and lower-temperature behaviours of low-lacunarity SC are different from the same behaviours obtained by hypercubic lattices of non-integer dimensionality (in the neighbourhood of $D = 2$).

The numerical results obtained by Bhanot *et al* (1984, 1985) are not conclusive about this point because they cannot obtain, in computer simulations, the low-lacunarity limit of the SC.

For the Ising model, Gefen *et al* (1983b) calculated, for $D \to 1$, the SC thermal critical exponent (within the K-type approximation) and showed that it coincides with the hypercubic lattice value. In this particular limit our results *confirm* their conclusion.

5. Conclusion

In this paper we have studied the critical behaviour of the q-state Potts model on Sierpinski carpets. We have constructed hierarchical lattices that have been shown to be adequate to simulate the SC. The critical frontiers of several SC are obtained for several values of q and the numerical results can certainly be considered as good approximations for the exact ones.

These hierarchical lattices enable us to study the low-lacunarity limit ($c \to \infty$) of the SC. Their high- and low-temperature-behaviour dominant terms of the two-site correlation function can be calculated and compared with the corresponding correlation function asymptotic behaviours of the analytical extension of hypercubic lattices to non-integer dimensionality D. We verify that these asymptotic behaviours (SC and analytical extension of hypercubic lattices) *coincide* when $D \to 1$ but the same *does not* happen when $D \to 2$. This fact leads us to believe that the low-lacunarity SC proposed by Gefen *et al* (1983b) may not, in general, be the correct geometric implementation of hypercubic lattices with non-integer dimensionality.

The limit $D \to 1$ (where both the first-order thermal critical exponent and the temperature asymptotic behaviours of the two-site correlation function of the SC and the hypercubic lattices with non-integer dimensionality coincide) certainly deserves special attention. The analysis of higher-order terms of the thermal critical exponent and correlation function could show whether or not the implementation is valid for this particular limit.

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